On the fundamental groups of compact Sasakian manifolds

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Abstract

We study the fundamental groups of compact Sasakian manifolds, which we call Sasaki groups. It is shown that all known Kähler groups are Sasaki, in particular, all finite groups are Sasaki. On the other hand, we show there exists many restrictions on the fundamental groups of compact Sasakian manifolds. We also study the Abel-Jacobi map of a compact Sasakian manifold and its applications to Sasaki groups.

1. Introduction

Recently Sasakian manifolds attract a lot of attention. See [BoG08] for a comprehensive introduction. However, very little is known for the fundamental groups of compact Sasakian manifolds. Here we make some steps on this direction.

Sasakian manifolds are odd dimensional analogue of Kähler manifolds. Recall a Riemannian manifold (M^{2n+1}, g) (We always consider connected manifolds in this paper) to be Sasakian if it has a unit Killing field ξ, satisfying the equation

$$R(X, \xi)Y = g < \xi, Y > X - g < X, Y > \xi$$

Given such a characteristic vector ξ (also called Reeb vector field), we define a $(1, 1)$ tensor ϕ to be given by $\phi(X) = \nabla_X \xi$ and the characteristic one form η to be given by $\eta(X) =< X, \xi >$. Altogether we call (g, ξ, η, ϕ) a Sasakian structure. The vector field ξ defines the characteristic foliation F_ξ with one dimensional leaves and the kernel of η denoted by D, called the contact bundle, inherits an almost complex structure by restriction of ϕ. Let $g^T = g - \eta \otimes \eta$. It turns out (g, ξ, η, ϕ) is a Sasakian structure iff $(D, g^T, \phi|_D, d\eta)$ defines a transversal Kähler structure with transversal Kähler form $d\eta$.

A Sasakian structure on M is called quasi regular if all leaves of characteristic foliation F_ξ are closed, otherwise it is called irregular. By a theorem

of Wadsley [Wa75], if a Sasakian structure on M is quasi regular, ξ generates a locally free S^1 action on M. A Sasakian structure on M is called regular if this action is free. If the Sasakian structure on M is quasi regular, then the quotient space M/F_ξ is a K$\ddot{a}$hler orbifold. In gerenal, there is no quotient space if it is irregular.

There is a natural transversal Levi-Civita connection ∇^T on M defined by $\nabla^T_X Y = [\nabla_X Y]^p$ if $X, Y \in D$ and $\nabla^T_\xi Y = [\xi, Y]^p$, $\nabla^T_X \xi = 0$, where we denote Z^p is the projection of Z to D for any $Z \in TM$ and ∇ is the usual Levi-Civita connection induced by g.

Now define the transversal curvature tensor R^T by

$$R^T(X,Y)Z = \nabla^T_X \nabla^T_Y Z - \nabla^T_Y \nabla^T_X Z - \nabla^T_{[X,Y]} Z.$$

$$R^T(X,Y,Z,W) = g < R^T(X,Y)Z, W > .$$

and the transversal sectional curvature of the plane spanned by X, Y is defined by

$$K^T(X,Y) = R^T(X,Y,Y,X)$$

where $X, Y, Z, W \in D$.

We say (M, g) has nonpositive transversal sectional curvature if and only if $K^T(X,Y) \leq 0$ for any linearly independent vectors $X, Y \in D$. Examples of compact Sasakian manifolds with nonpositive transversal sectional curvature are given by Boothby-Wang fibrations over compact locally Hermitian symmetry spaces of noncompact type. See section 2 for details on Boothby-Wang fibration.

We say a finitely generated group is Sasaki (K$\ddot{a}$hler) if it is the the fundamental group of a compact Sasakian (K$\ddot{a}$hler) manifold.

It is known by various geometers that any compact Sasakian manifold has even first Betti number, see [BG67], [F66], [Tan67]. Since one can lift a Sasakian structure to the covering space of a Sasakian manifold, the following proposition is immediate.

Propositon 1.1 *A group which contains a finite index subgroup with odd first Betti number is not Sasaki. In particular, nontrivial free groups can not be Sasaki.*

The next proposition gives many examples of Sasaki groups.

Propositon 1.2 *If Γ is the fundamental group of some compact Hodge manifold, it is also Sasaki. In partiular, all known Kähler groups are Sasaki.*

Using a classic result of Serre ([ABCKT96]) that any finite group is the fundamental group of some compact Hodge manifold and also propositions 1.1, 1.2, one easily gets

Corollary 1.3 *All finite groups are Sasaki and an abelian group is Sasaki if and only if it has even rank.*

It was conjectured that any Kähler group is the fundamental group of some compact Hodge manifold. So it is natural to propose the following

Conjecture *All Kähler groups are Sasaki.*

It is easy to see there exists Sasaki groups which are not Kähler. For example, the discrete, torsion free and cocompact subgroups of real Heisenberg group H^{2n+1} are Sasaki. However, they are not Kähler if $n \leq 3$ by a theorem of Carlson and Toledo, see [CaT95].

The next theorem tells us some well known results for Kähler groups remain true for Sasaki groups.

Theorem 1.4 *Suppose* Γ *is a Sasaki group, then*
(1) Γ *has either* 0 *or* 1 *end; in particular,* Γ *can not split as a nontrivial free product;*
(2) If Γ *is solvable, it contains a nilpotent subgroup of finite index.*

The following proposition gives more restrictions on the fundamental groups of compact regular Sasakian manifolds.

Proposition 1.5 *Suppose* Γ *is the fundamental group of some compact regular Sasakian manifold. Then*
(1) Γ *can not be a cocompact, discrete and torsion free subgroup of* G*, where* G *is* $SO(1,n), n > 2$ *or* $F_{4(-20)}$ *or a simple real non-Hermitian Lie group of noncompact type with real rank at least 20.*
(2) Γ *is the fundamental group of some compact* 3*-manifold* V^3 *if and only if the universal cover of* V^3 *is differemorphic to* S^3 *or the* 3*-dimensional Heisenberg group or* $\widetilde{SL(2,\mathbb{R})}$*.*

It is quite believable that proposition 1.5 holds for all Sasaki groups without any regular assumption.

Theorem 1.6 *Suppose (M^{2n+1}, g) is a compact Sasakian manifold with nonpositive transversal sectional curvature, then $\pi_1(M^{2n+1})$ is an infinite group which can not be realized as the fundamental group of any compact Riemannian manifold with nonpositive sectional curvature.*

As a consequence of theorem 1.6, we recover the following fact which was previous known by P. Eberlein's splitting theorem ([Eb82]): If $\Gamma = \pi_1(M^3)$, where M^3 is a compact 3-manifold with geometry modeled on $\widetilde{SL(2,\mathbb{R})}$, Γ can not be the fundamental group of any compact Riemannian manifold with nonpositive sectional curvature. However, as far as the author konws, theorem 1.6 does not follow from P. Eberlein's theorem if $n > 1$.

Theorem 1.6 suggests to study a new class of infinite groups, which we hope to cover in the future.

The organization of this paper is the following: In section 2 we review two constructions in Sasakian geometry and prove proposition 1.2. Theorem 1.4 is proved in section 3. In section 4 we study harmonic maps from compact Sasakian manifolds and prove proposition 1.5. In section 5 we prove theorem 1.6. In section 6 we study the Abel-Jacobi map of a compact Sasakian manifold and its applications to Sasaki groups.

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2. Constructions in Sasakian geometry and proof of proposition 1.2

We first recall two basic constructions in Sasakian geometry. For details, see [BoG08]. For simplicity, we restrict ourselves to the class of regular Sasakian manifolds.

Boothby-Wang fibration: Suppose (N^{2n}, ω) is a compact Hodge manifold

with integral K$\ddot{a}$hler form ω. By a theorem of Kobayashi [Kob63], there exists a principal circle bundle P over N^{2n} and a connection form η on P such that $d\eta = p^*\omega$, where p is the projection map. Let M^{2n+1} be the total space of P. Now define a Riemanian metric g on M^{2n+1} by $g = p^*h + \eta \otimes \eta$, where h is the associated K$\ddot{a}$hler metric for ω on N^{2n}. It is not hard to check that (M^{2n+1}, g) becomes a Sasakian manifold with transversal K$\ddot{a}$hler form ω. We call (M^{2n+1}, g) is the Boothby-Wang fibration over (N^{2n}, h). It is not hard to extend this construction to the case when (N^{2n}, ω) is a compact Hodge orbifold.

From the construction of Sasakian structure on M^{2n+1}, it is easy to see (M^{2n+1}, g) has nonpositive transversal sectional curvature if (N^{2n}, h) has nonpositive sectional curvature, for example, (N^{2n}, h) is a compact locally Hermitian symmetry space of noncompact type.

Join construction: Suppose M_1 and M_2 are two compact Sasakian manifolds over compact Hodge manifolds (N_1, ω_1) and (N_2, ω_2), respectively. Then $(N_1 \times N_2, \omega_1 + \omega_2)$ is also a compact Hodge manifold. Denote $M_1 * M_2$ be the Sasakian manifold over $N_1 \times N_2$ coming from Boothby-Wang fibration. We call $M_1 * M_2$ is the join of M_1 and M_2. It turns out $M_1 * M_2$ is a M_2 bundle over N_1, see [BoG08]. It is easy to see compact Sasakian manifolds with nonpositive transversal sectional curvature are closed under join construction.

Proof of proposition 1.2

Suppose $\Gamma = \pi_1(N)$, where N is a compact Hodge manifold. Let M be the Sasakian manifold over N coming from Boothby-Wang fibration. Let $V = M * S^3$. Then V is a compact Sasakian manifold and also a S^3 bundle over N. From the long exact sequence of homotopy groups, one gets $\pi_1(V) \simeq \Gamma$.

3. Orbifold fundamental groups of compact K$\ddot{a}$hler orbifolds

We refer the readers to [Da08] or [Th78] for general notions on orbifolds and orbifold fundamental groups.

In [C11], the author studied the orbifold fundamental groups of K$\ddot{a}$hler orbifolds. In particular, he proved the following:

Lemma 3.1 *Suppose G is the orbifold fundamental group of a compact Kähler*

orbifold, then
(1) G has 0 or 1 end;
(2) If G is solvable, it contains a nilpotent subgroup of finite index.

Remark: The first part of lemma 3.1 is not explicitly stated in [C11], however, it follows from the arguments there and the corresponding result in the manifold case.

We recall some basic facts on the end of a finitely generated group. See [DK] for more details. Suppose X is a locally compact connected topological space. The set of ends of X, denoted by $E(X)$, is defined as the inverse limit:

$$\lim_{K\subseteq X} \pi_0(K^c)$$

, where K is a compact subset of X, K^c is the compliment of K and $\pi_0(K^c)$ is the number of path-connected components of K^c. If Γ is a finitely generated group, the space of ends $E(\Gamma)$ is defined as the set of ends of its Cayley graph. The elements in $E(\Gamma)$ are called the ends of Γ. We denote $e(\Gamma)$ be the cardinality of $E(\Gamma)$. It can be shown $e(\Gamma)$ is a quasi-isometric invariant of Γ. The following facts are due to Hopf and Freudenthal:

1. $e(\Gamma) \in \{0, 1, 2, \infty\}$.
2. $e(\Gamma) = 0$ if and only if Γ is finite.
3. $e(\Gamma) = 2$ if and only if Γ has an infinite cyclic subgroup of finite index.

It is a well known theorem of Stalling that a finitely generated group has more than one end if and only if it is a nontrivial amalgamated free product or an HNN extension over a finite subgroup, see [Be68], [St68], [St71].

Lemma 3.2 *Suppose we have the following short exact squence of groups*

$$1 \to \mathbb{Z} \to A_1 \to A_2 \to 1$$

then A_1 has 1 or 2 ends.

Proof. If A_2 is finite, A_1 contains an infinite cyclic subgroup of finite index and hence has 2 ends. Otherwise A_2 is infinite, it follows that A_1 has 1 end by proposition 1.9 in [Co70]. $\square$

Lemma 3.3 *Suppose we have the following short exact squence of groups*

$$1 \to B_1 \to B_2 \to B_3 \to 1$$

and B_1 is a cyclic group and B_3 is virtually nilpotent, then B_2 is virtually nilpotent. Here we say a group is virtually nilpotent if it contains a nilpotent subgroup of finite index.

Proof. By passing to a subgroup of finite index, we can assume B_3 is a nilpotent group. Since B_1 is normal in B_2, we see B_2 acts on B_1 by conjugation. This gives a group homomorphism from B_2 to $Aut(B_1)$. Since B_1 is a cyclic group, $Aut(B_1)$ is a finite group. Hence up to a subgroup of finite index, we get a central group extension

$$1 \to B_1 \to B_2 \to B_3 \to 1$$

where B_1 is cyclic and B_3 is nilpotent. Now it is easy to see B_2 is also nilpotent. $\square$

Now we use the above lemmas to prove theorem 1.4. In fact, assume $\Gamma \simeq \pi_1(M)$, where M is a compact Sasakian manifold. By a theorem of Wadsley [Wa75], we can always assume M is quasi-regular. So we can assume M is an orbifold S^1 bundle over some compact Kähler orbifold N and $p : M \to N$ is the projection map. Then we have the following long exact sequence

$$\to \pi_2^{orb}(N) \to \mathbb{Z} \to \pi_1(M) \to \pi_1^{orb}(N) \to 1$$

From which we get the following short exact sequence

$$1 \to \Gamma_1 \to \pi_1(M) \to \pi_1^{orb}(N) \to 1$$

where Γ_1 is a cyclic group.

Now we prove the first part of theorem 1.4. First note Γ can not have 2 ends otherwise it contains an infinite cyclic subgroup of finite index which is impossible by proposition 1.1. If Γ_1 is an infinite cyclic group, Γ has 1 end by lemma 3.2. Otherwise Γ_1 is finite and Γ is quasi-isometric to $\pi_1^{orb}(N)$. It follows that Γ has 0 or 1 end by lemma 3.1.

If Γ is solvable, then $\pi_1^{orb}(N)$ is also solvable. It follows that $\pi_1^{orb}(N)$ is a virtually nilpotent group by lemma 3.1 and Γ is also virtually nilpotent by lemma 3.3. This proves the second part of theorem 1.4.

4. Harmonic maps from compact Sasakian manifolds

We first state the following theorem due to R. Petit. See [Pet02].

Theorem 4.1 *Suppose f is a harmonic map from a compact Sasakian manifold (M_1, h_1) to a compact Riemannian manifold (M_2, h_2) with nonpositive sectional curvature, then $f_*(\xi) = 0$, where ξ is the Reeb vector field associated to M_1. Moreover, if M_1 is a regular Sasakian manifold fibering over Kähler manifold N and $p : M_1 \to N$ is the projection map, there exists a harmonic map g from N to M_2 such that $f = gp$.*

It was known by C.Boyer and K. Galicki [BoG08] that the connected sum of two compact negatively curved manifolds can not admit Sasakian structure. The following corollary is a generalization of this result.

Corollary 4.2 *There is no continous map of nonzero degree from a compact Sasakian manifold to the connected sum $M_1^n \sharp M_2^n$, where (M_1^n, g_1) is a compact Riemannian manifold with nonpositive sectional curvature and M_2^n is any compact n-dimensional manifold. In particular, $M_1^n \sharp M_2^n$ can not admit Sasakian structure.*

Proof. Suppose f_1 is a continous map of nonzero degree from a compact Sasakian manifold (M_0^n, g_0) to $M_1^n \sharp M_2^n$. Note there is a continous map of degree one $f_2 : M_1^n \sharp M_2^n \to M_1^n$. Let $f_3 = f_2 f_1$, then $f_3 : (M_0^n, g_0) \to (M_1^n, g_1)$ is a continous map of nonzero degree. By a classic theorem of Eells and Sampson ([ES64]), within the same homotopy class, we can find a harmonic representive f_4. On the other hand, by theorem 4.1, f_4 can not be surjective by Sard's theorem. It follows that the degree of f_4 is zero. Contradiction. $\square$

As an application of corollary 4.2, any compact manifold admitting a metric of nonpositive sectional curvature cannot admit Sasakian structure. For example, suppose H^3 is a 3-dimensional hyperbolic homology sphere and T^2 is the 2-dimensional torus. Then $T^2 \times H^3$ can not admit Sasakian structure by corollary 4.2. However, as far as the author knows, the nonexistence of Sasakian structure on $T^2 \times H^3$ does not follow from any previously known obstructions.

Now we are about to prove proposition 1.5. For the rest of this section, we always assume M is a compact regular Sasakian manifold fibering over some compact Kähler manifold N and $p : M \to N$ is the projection map. Denote $\Gamma = \pi_1(M)$.

We first show the first part. Now assume $\Gamma \simeq \pi_1(B)$, where $B = \Gamma \setminus G/K$, where K is a maximal compact subgroup of G. It follows that B is a compact locally symmetry space of noncompact type and so admits a

Riemannian metric of nonpositive sectional curvature. By a classic theorem of Eells and Sampson, we can choose a harmonic map $f : M \to B$ inducing the isomorphism between $\pi_1(M)$ and $\pi_1(B)$. By theorem 4.1, there exists a harmonic map g from N to B such that $f = gp$. It follows that $p_* : \pi_1(M) \to \pi_1(N)$ is injective. On the other hand, from the long exact sequence of homotopy groups we know p_* is surjective and hence p_* is an isomorphism. Now Γ is a Kähler group and also a cocompact lattice in G, where G is $SO(1,n), n > 2$ or $F_{4(-20)}$ or a simple real non-Hermitian Lie group of noncompact type with real rank at least 20. This is impossible by the results of Carlson, Hernández, Klingler and Toledo. See [CaT89], [CaH91], [Kl10].

Now we prove the second part. Suppose $\Gamma = \pi_1(M) \simeq \pi_1(V^3)$, where V^3 is a compact 3-dimensional manifold. By passing to orientable cover, we can assume V^3 is orientable. First we have the following long exact sequence of homotopy groups:

$$\to \pi_2(N) \to \pi_1(S^1) \to \pi_1(M) \to \pi_1(N) \to 1$$

From it we get a short exact sequence:

$$1 \to i_*(\pi_1(S^1)) \to \pi_1(M) \to \pi_1(N) \to 1$$

where $i : S^1 \to M$ is the inclusion map. If $i_*(\pi_1(S^1)) = \{1\}$, $\Gamma = \pi_1(M) \simeq \pi_1(N)$ and so Γ is a Kähler group. By assumption, we have $\Gamma \simeq \pi_1(V^3)$. It follows that Γ is finite by a theorem of A. Dimca and A. Suciu, see [DS09] and [Kot11]. Hence the universal cover of V^3 is differemorphic to S^3 by the work of Perelman [Per02], [Per03]. Now we assume $i_*(\pi_1(S^1))$ is a nontrivial cyclic group. Then $\pi_1(V^3)$ contains a nontrivial cyclic normal subgroup. In this case, we first

Claim: *V^3 is a Seifert manifold.*

Given the above claim, it follows that V^3 carries one of the geometries $S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \mathbb{R}^3, S^3, \widetilde{SL(2,\mathbb{R})}$, Nil, where Nil is the 3-dimensional Heisenberg group. But those manifolds carrying one of the geometries $S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \mathbb{R}^3$ have virtually odd first Betti number. However, any Sasaki group has even first Betti number by proposition 1.1. So the universal cover of V^3 is differemorphic to S^3 or the 3-dimensional Heisenberg group or $\widetilde{SL(2,\mathbb{R})}$. The other direction is obvious.

Proof of claim

First of all, we can assume V^3 is prime in the sense being indecomposable under connected sum, since a nontrivial free product is never a Sasaki group by theorem 1.4. By assumption, $\pi_1(V^3)$ contains a nontrivial cyclic normal subgroup Γ_0. Firstly suppose Γ_0 is a finite cyclic group. Then $\pi_1(V^3)$ has nontrivial torsion. Since V^3 is a prime also orientable 3-manifold, it follows that $\pi_1(V^3)$ must be finite by a theorem of Epstein, see [Ep61], [H76]. So the universal cover of V^3 is differemorphic to S^3 by the work of Perelman. Now we assume Γ_0 is an infinite cyclic group. Since V^3 is prime and orientable, it is either $S^2 \times S^1$ or irreducible, that is, every embedded S^2 bounds a 3-cell. Because a Sasaki group can not be infinite cyclic by corallary 1.3, we see V^3 must be irreducible. On the other hand, $\pi_1(V^3)$ contains an infinite cyclic normal subgroup and so V^3 must be a Seifert manifold by a theorem of A. Casson, D. Jungreis and D. Gabai, see [CaJ94], [Ga92].

5. A transversal Jacobi equation and proof of theorem 1.6

The proof of theorem 1.6 is based on the following transversal Cartan-Hadamard theorem:

Theorem 5.1 *Suppose (M^{2n+1}, g) is a complete Sasakian manifold with nonpositive transversal sectional curvature, then its universal cover is differemorphic to $\mathbb{R}^{2n+1}$.*

Corollary 5.2 *Suppose (M^{2n+1}, g) is a compact Sasakian manifold with nonpositive transversal sectional curvature, then its fundamental group can not be Gromov hyperbolic.*

In fact, by a theorem of I. Mineyev [Mi01], [Mi02], any compact aspherical manifold has positive simplicial volume if its fundamental group is Gromov hyperbolic. Since any compact Sasakian manifold has vanishing simplicial volume, corollary 5.2 follows.

By the classification of compact Sasakian manifolds in dimension 3 ([BoG08]), we get

Corollary 5.3. *Suppose (M^3, g) is a 3-dimensional compact Sasakian manifold with nonpositive transversal sectional curvature, then M^3 is differemorphic to the*

compact quotient of the 3*-dimensional Heisenberg group or* $\widetilde{SL(2,\mathbb{R})}$.

On the other hand, compact quotients of the 3-dimensional Heisenberg group ($\widetilde{SL(2,\mathbb{R})}$) are Seifert circle bundles over flat (hyperbolic) orbifolds and so admit Sasakian structure with nonpositive transversal sectional curvature.

Before we prove theorem 5.1, we show how to derive theorem 1.6 from it. We prove it by contradiction. Suppose $\Gamma = \pi_1(M, g) \simeq \pi_1(N, h)$, where (M, g) is a compact Sasakian manifold with nonpositive transversal sectional curvature and (N, h) is a compact manifold with nonpositive sectional curvature. Then by a classic result of Eells and Sampson, there exists a harmonic map f which induces the isomorphism of the fundamental groups. By a theorem of A. Banyaga and P. Rukimbira ([Ban90], [Ruk95], [Ruk99]), there exists a closed leaf of the characteristic foliation on M. Since M is compact, we see this leaf is differemorphic to S^1. From the proof of theorem 5.1, we see this S^1 is lifted to a real line under covering map $p : \widetilde{M} \to M$, where $\widetilde{M}$ is the universal cover of M. This implies the inclusion map $i_* : \pi_1(S^1) \to \pi_1(M)$ is injective. On the other hand, since f is harmonic, using theorem 4.1, $f_*(\xi) = 0$, where ξ is the Reeb vector field associated to M. So f maps a nontrivial subgroup of Γ to zero, which contradicts that f is an isomorphism.

Now we are in position to prove theorem 5.1. The idea is similar to the proof of Cartan-Hadamard theorem. Suppose $\widetilde{M^{2n+1}}$ is the universal cover of M^{2n+1} . Then it is also a complete Sasakian manifold with nonpositive transversal sectional curvature. Suppose $\alpha(s)$ is a leaf of the characteristic foliation F_ξ on $\widetilde{M^{2n+1}}$. We show its normal exponential map $exp^\perp : \alpha(s)^\perp \to \widetilde{M^{2n+1}}$ is a differmorphism, where $\alpha(s)^\perp$ is the normal bundle of $\alpha(s)$. To do this, suppose $\gamma(t)$, $t \in [0, 1]$ is any minimizer geodesic which is perpendicular to $\alpha(s)$ at $\alpha(0) = \gamma(0)$ and $J(t)$ is any Jacobi field along $\gamma(t)$ such that $J(0) = \lambda\xi(0)$ and $J(1) = 0$, where $\xi(0) = \alpha'(0)$. It suffices to show $J(t) \equiv 0$. Then we see there is no focal point to $\alpha(s)$ and so $exp^\perp$ is a differmorphism.

There is a natural splitting $T_{\gamma(t)}M = \xi(t) \oplus \xi(t)^\perp$, where $\xi(t)$ is the tangential part to leaves of the characteristic foliation F_ξ and $\xi(t)^\perp$ is the orthogonal part. Following B.Wilking [Wi07], define $Y(t) = J^\perp(t)$ and $\nabla^\perp_{\frac{\partial}{\partial t}} Y = (\nabla_{\frac{\partial}{\partial t}} Y)^\perp = (\nabla_{\dot\gamma} Y)^\perp$, where we denote $X^\perp(t)$ is the projection of $X(t)$ to $\xi(t)^\perp$ for any vector field $X(t)$ along $\gamma(t)$.

Lemma 5.4 $Y(t)$ satisfies the following transversal Jacobi equation:

$$\nabla^{\perp}_{\frac{\partial}{\partial t}}\nabla^{\perp}_{\frac{\partial}{\partial t}}Y + R^T(Y,\dot{\gamma})\dot{\gamma} = 0$$

Before we prove it, we need the following lemma.
Lemma 5.5

$$R^T(X,Y)Y = (R(X,Y)Y)^{\perp} + 3 < X, \phi(Y) > \phi(Y), where X, Y \perp \xi.$$

Proof. Choose any vector field Z such that $Z \perp \xi$. We see

$$R^T(X,Y,Y,Z) =< R^T(X,Y)Y, Z >=< \nabla^T_X\nabla^T_Y Y - \nabla^T_Y\nabla^T_X Y - \nabla^T_{[X,Y]}Y, Z >$$

$$=< \nabla_X(\nabla_Y Y - < \nabla_Y Y, \xi > \xi) - \nabla_Y(\nabla_X Y - < \nabla_X Y, \xi > \xi)$$

$$-\nabla^T_{[X,Y]-<[X,Y],\xi>\xi}Y - \nabla^T_{<[X,Y],\xi>\xi}Y, Z >$$

$$=< R(X,Y)Y, Z > + < \nabla_X Y, \xi >< \nabla_Y \xi, Z > + < [X,Y], \xi >< \nabla_Y \xi, Z >$$

$$=< R(X,Y)Y, Z > +3 < X, \phi(Y) >< \phi(Y), Z >$$

where the last equality follows from $X, Y \perp \xi$ and ξ is a Killing vector field. □

Now we are in position to prove lemma 5.4. It suffices to prove it at generic t_0, i.e. $J(t_0) \neq 0$. First note we can assume $Y(t_0) = J(t_0)$. Choose vector fileds $X_i(t), i = 1, 2, \cdots, 2n$ such that $X_i \perp \xi$ and also $J(t_0) = X_1(t_0), \nabla^{\perp}_{\frac{\partial}{\partial t}}X_i(t) = 0, < X_i, X_j > (t) = \delta_{ij}$ for all $t \in [0,1]$. First note we have

$$X_i' =< X_i', \xi > \xi =< \nabla_{\dot{\gamma}}X_i, \xi > \xi = - < X_i, \nabla_{\dot{\gamma}}\xi > \xi$$

Using this, at t_0, we have

$$< \nabla^{\perp}_{\frac{\partial}{\partial t}}\nabla^{\perp}_{\frac{\partial}{\partial t}}Y, X_i >=< \nabla_{\frac{\partial}{\partial t}}\nabla^{\perp}_{\frac{\partial}{\partial t}}Y, X_i >$$

$$= \frac{\partial}{\partial t} < \nabla^{\perp}_{\frac{\partial}{\partial t}}Y, X_i > - < \nabla^{\perp}_{\frac{\partial}{\partial t}}Y, \nabla^{\perp}_{\frac{\partial}{\partial t}}X_i >$$

$$= \frac{\partial}{\partial t} < \nabla^{\perp}_{\frac{\partial}{\partial t}}Y, X_i >= \frac{\partial^2}{\partial t^2} < J, X_i >$$

$$=< J'', X_i > +2 < J', X_i' > + < J, X_i'' >$$

$$= - < R(Y,\dot{\gamma})\dot{\gamma}, X_i > +2 < J', X_i' > + < X_1, X_i'' >$$

$$= - < R(Y,\dot{\gamma})\dot{\gamma}, X_i > +2 < J', X_i' >$$

$$+\frac{d}{dt}_{t=t_0} < X_1, X_i' > - < X_1', X_i' >$$

$$= - < R(Y,\dot\gamma)\dot\gamma, X_i > +2 < J', X_i' > - < X_1', X_i' >$$

$$= - < R(Y,\dot\gamma)\dot\gamma, X_i > -2 < J', \xi >< X_i, \xi' > - < X_1, \xi' >< X_i, \xi' >$$

$$= - < R(Y,\dot\gamma)\dot\gamma, X_i > -2 < J, \xi' >< X_i, \xi' > - < J, \xi' >< X_i, \xi' >$$

$$=< -R^T(Y,\dot\gamma)\dot\gamma, X_i >$$

where the last equality follows from lemma 5.5 and the last second equality follows from

Lemma 5.6

$$< J'(t), \xi(t) >=< J(t), \xi'(t) >$$

Proof.

$$< J'(t), \xi(t) >=< \nabla_{\dot\gamma} J, \xi > (t)$$

$$=< \nabla_J \dot\gamma, \xi > (t) = - < \dot\gamma, \nabla_J \xi > (t)$$

$$=< J, \nabla_{\dot\gamma} \xi > (t) =< J(t), \xi'(t) >$$

$\square$

Now define $f(t) = \frac{1}{2}\|Y(t)\|^2$, then $f(0) = f(1) = 0$ since $Y(0) = Y(1) = 0$. Moreover, we have

$$f'(t) =< \nabla^{\perp}_{\frac{\partial}{\partial t}} Y, Y >$$

and also

$$f''(t) =< \nabla^{\perp}_{\frac{\partial}{\partial t}} Y, \nabla^{\perp}_{\frac{\partial}{\partial t}} Y > + < \nabla^{\perp}_{\frac{\partial}{\partial t}} \nabla^{\perp}_{\frac{\partial}{\partial t}} Y, Y >$$

$$=< \nabla^{\perp}_{\frac{\partial}{\partial t}} Y, \nabla^{\perp}_{\frac{\partial}{\partial t}} Y > -R^T(Y,\dot\gamma,\dot\gamma,Y) \geq 0$$

From this we know $f(t)$ is a convex function with $f(0) = f(1) = 0$ and so $f(t) = 0$ for any $t \in [0,1]$. So $Y(t) \equiv 0$ and we can assume $J(t) = \lambda(t)\xi(t)$ with $\lambda(0) = \lambda$. Then $\lambda(t) =< J(t), \xi(t) >$ and $\lambda(1) = 0$ since $J(1) = 0$. Taking derivative with respect to t, we get

$$\lambda'(t) =< J'(t), \xi(t) > + < J(t), \xi'(t) >$$

$$= 2 < J(t), \xi'(t) >= 2 < \lambda(t)\xi(t), \xi'(t) >= 0.$$

so $J(t) \equiv 0$ and $exp^{\perp} : \alpha(s)^{\perp} \to \widetilde{M^{2n+1}}$ is a differmorphism. Hence $\widetilde{M^{2n+1}}$ is diferemorphic to the normal bundle of a 1-dimensional manifold. Since $\widetilde{M^{2n+1}}$ is simply connected, we see $\widetilde{M^{2n+1}}$ is differemorphic to $\mathbb{R}^{2n+1}$.

6. Abel-Jacobi maps of compact Sasakian manifolds

We first recall the construction of Abel-Jacobi map of any compact Riemannian manifold, see [Ka07] for more details.

Suppose (M, g) is a compact Riemannian manifold. Let $\pi = \pi_1(M)$ be its fundamental group. Set $f : \pi \to \pi^{ab}$ be the abelianisation map and $g : \pi^{ab} \to \pi^{ab}/tor$ be the quotient by torsion. Suppose $\bar{M}$ is the covering space of M with $\pi_1(\bar{M}) = ker(\phi)$, where $\phi = gf$.

Let E be the space of harmonic 1-form on M, with dual E^* canonically identified with $H_1(M, \mathbb{R})$. Fix a basepoint $x_0 \in M$. Then any point x in the universal cover $\tilde{M}$ of M is represented by a point of M together with a path c from x_0 to it. By integrating along the path c, we get a linear form, $h \to \int_c h$, on E. We thus obtain a map $\tilde{M} \to E^* = H_1(M, \mathbb{R})$, which descends to a map

$$\bar{A_M} : \bar{M} \to E^*, c \mapsto (h \mapsto \int_c h),$$

By definition, the Jacobi torus of M is the torus

$$J_1(M) = H_1(M, \mathbb{R})/H_1(M, \mathbb{Z})_{\mathbb{R}}$$

and the Abel-Jacobi map

$$A_M : M \to J_1(M),$$

is obtained from the map $\bar{A_M}$ by passing to quotients. From the construction, it is not hard to see the Abel-Jacobi map induces an isomorphism between the first homology groups with real coefficient.

Proposition 6.1 *Suppose (M, g) is a compact Sasakian manifold and F_ξ is the characteristic foliation on M. Then the restriction of the Abel-Jacobi map of M to any leaf of F_ξ is a constant map.*

Proof. By a theorem of Tachibana ([Tac65]), any harmonic one form h on M satisfies $h(\xi) = 0$. Using it, proposition 6.1 easily follows from the construction of Abel-Jacobi map. □

Corollary 6.2 *Suppose (M, g) is a compact Sasakian manifold and $i : S^1 \to M$ is the inclusion map, where S^1 is a closed leaf of the characteristic foliation F_ξ on M. Then $i_*(\pi_1(S^1)) \subseteq ker\phi$, where ϕ is the map constructed in the beginning of this section. In other words, S^1 generates a trivial element in $H_1(M, \mathbb{R})$.*

Proof. By proposition 6.1, the Abel-Jacobi map of M takes S^1 to a point. However, the Abel-Jacobi map induces an isomorphism between the first homology groups with real coefficient and so S^1 generates a trivial element in $H_1(M, \mathbb{R})$. $\square$

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